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TIME MARCHING NUMERICAL SOLUTION OF THE DYNAMIC RESPONSE OF NONLINEAR SYSTEMS

by

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National Aeronautical Establishment

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SOLUTION NUMÉRIQUE TEMPORELLE AU PROBLÈME DE LA RÉPONSE DYNAMIQUE DES SYSTÈMES NON-LINÉAIRES

by/par

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SUMMARY

As part of the study of the nonlinear response of structural components we investigate in this report a single degree-of-freedom nonlinear response equation usually known as Duffing's equation. Previously the response curves have been investigated analytically by many authors but there seems to have been very little numerical work done in the nonlinear cases. It is shown that a numerical solution is feasible and that some of the standard analytical approximations are not accurate in the peak region of the response curve.

RÉSUMÉ

Dans le cadre d'une étude de la réponse non-linéaire d'éléments structuraux, les auteurs analysent une équation à un seul degré de liberté relative à la réponse non-linéaire, connue généralement sous le nom d'équation de Duffing. Les courbes de réponse ont déjà été analysées par de nombreux auteurs, mais il semble qu'on ait fait très peu de travaux de nature numérique sur les cas non-linéaires. Les auteurs démontrent qu'une solution numérique est possible et que certaines des approximations analytiques standard ne sont pas exactes en ce qui concerne le pic de la courbe de réponse.

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Symbols

A	final amplitude after transients have died down
A_2, A_3, A_4	constants in linear solution
c	damping coefficient
F	amplitude of driving force
IMAX, IMIN	step number at which maximum or minimum x occurs
k	stiffness
N	number of steps per cycle of period $\frac{2\pi}{\omega}$
t	time
x	displacement variable describing the motion
β	coefficient of x^3 in the nonlinear equation
Δt	time step in the integration process
ζ	damping ratio = $c/2\sqrt{k}$
ρ	spectral radius of the stability matrix
ω	amplitude of driving force

Subscripts

o	initial conditions
n	at time t or $n\Delta t$

1.0 INTRODUCTION

In recent years there have been rapid developments in structural analysis techniques and related computer codes. The analysis of linear response of structural components has been well studied and numerous numerical time marching techniques have been developed for finite element structural analysis programs. Some of the more commonly used time integration techniques are the explicit central difference scheme and Houbolt's, Wilson's and Newmark's methods (Ref. 1). In order for any algorithm to be useful in practice, it must not lead to a divergent solution. Except for the explicit scheme, the other three methods are unconditionally stable. A study of the accuracies shows that all three schemes suffer from a period elongation which increases with increase in time step. In addition, Houbolt's and Wilson's methods have artifical damping resulting in the higher modes being rapidly damped out in a system with a large number of degrees of freedom.

In non-linear problems established methods for investigating stability and decay as in linear responses are not possible. Various formulations have been proposed and used with varying degrees of success. The four methods described in [1] for linear problems can readily be used for the nonlinear case. The choice of any particular algorithm depends to a large extent on the type of problem to be solved.

The use of time marching techniques in flutter studies was first reported by Ballhaus and Goorjian [2] who carried out the aeroelastic response of a NACA 64A006 airfoil with a single

degree-of-freedom in pitch at transonic speeds. Extensions of this procedure to two- and three-degree-of-freedom have been reported by Rizzetta [3] and Yang and Chen [4]. Only a linear response was treated by these authors. In flutter studies, there are potentially many sources of non- linearities present, some of which may give rise to "limit-amplitude" oscillations. The nonlinearities may arise in many possible ways, but the most commonly encountered ones are those having structural or aerodynamic origin.

In aeroelastic applications unconditionally stable methods like Houbolt's for linear problems is not very critical, since the number of modes considered is usually not very large. Conditionally stable schemes can then be used, provided a time step is chosen to ensure the highest mode does not diverge. Also, it is important that amplitude decay and period elongation should be as small as possible. In the present study higher order implicit finite difference schemes with greater accuracies than Houbolt's method are discussed. To study the stability and accuracies of the higher order schemes, a linear system is considered.

In the application to the nonlinear case the restoring force is assumed to be given by that of a cubic spring. The equation to be solved is essentially Duffing's equation [5]. In aeroelastic problems a cubic restoring force corresponding to a hardening spring arises when a thin wing or propeller is subjected to increasingly large amplitudes in torsion.

The numerical solution to the nonlinear Duffing's equation offered some interesting studies. For instance, it was not known how

a multivalued solution could be computed and whether the jump phenomenon from one branch of the response curve to another could be forecast numerically. What would affect such a jump and would the jump be consistent? Also would the unstable part of the response curve cause numerical difficulties? Are the approximate analytic response curves accurate?

We present here solutions which answer the above questions and hopefully this will clear the way to study higher order degrees-of-freedom systems and to more practical applications.

2.0 THE RESPONSE EQUATIONS

2.1 The Linear Equation

In order to study our numerical schemes we first consider the linear equation for damped oscillations i.e.

$$\ddot{x} + c\dot{x} + kx = F \sin \omega t$$

Its solution is

$$x = e^{-ct/2} \left[A_2 \sin \sqrt{(1-\zeta^2)}\, \omega_n t + A_3 \cos \sqrt{(1-\zeta^2)}\, \omega_n t\right] + A_4 \sin(\omega t - \phi)$$

where

$$A_3 = x_0 + A_4 \sin\phi$$

$$A_2 = \left[\dot{x}_0 + A_3\, \zeta\omega_n - A_4 \omega \cos\phi\right] / \sqrt{(1-\zeta^2)}\, \omega_n$$

$$A_4 = \frac{F}{\omega_n^2} \left[(1-\Omega^2)^2 + (2\zeta\Omega)^2\right]^{-\frac{1}{2}}$$

$$\zeta = \frac{c}{2\sqrt{k}} \; ; \; \omega_n = \sqrt{k} \; ; \; \Omega = \frac{\omega}{\omega_n}$$

$$\tan\phi = \frac{2\zeta\Omega}{1-\Omega^2}$$

and x_0, $\dot{x}_0$ are the initial displacement and velocity respectively. It can be seen that transients represented by the first term for x die down and leave the particular integral $A_4 \sin(\omega t-\phi)$. Thus the initial conditions x_0, $\dot{x}_0$ are finally unimportant in this linear case. Only in the nonlinear system will they be critical as shown by the numerical solutions.

2.2 The Nonlinear Equation

Many articles have been written on the equation which represents forced oscillations of systems with a nonlinear restoring force. The equation can be written

$$\ddot{x} + c\dot{x} + f(x) = F \sin \omega t$$

for $c > 0$. Most of the studies have centred on displacements large enough to include linear and cubic terms in f(x) i.e. the equation

$$\ddot{x} + c\dot{x} + kx + \beta x^3 = F \sin \omega t \qquad (1)$$

is normally considered and this is the form we will concentrate upon in this study. This equation is normally referred to as Duffing's equation, since Duffing [5] was the first to make significant progress in studying it.

Duffing in his study neglected the damping force $c\dot{x}$ and, by an expansion method, found that to first order the equation

$$\omega^2 = k + \frac{3}{4}\beta A^2 - \frac{F}{A} \qquad (2)$$

related ω to the amplitude A of the oscillation. The equation (2) is an interesting relation in that for certain values of the parameters there are three values of A associated with one value of ω as illustrated in Fig. 1. This gave rise to speculation that a jump phenomenon might exist in which the solution would suddenly jump from

one branch 5-4-6 say in Fig. 1 to the branch 1-2-3 with the branch 3-6 being unstable. A hysteresis effect may possibly be observed if the amplitude follows the path 5-4-6-2-1 for increasing ω and the path 1-2-3-4-5 for decreasing ω. Figure 1 shows the situation for $\beta > 0$ (hard spring i.e. the stiffness increases with displacement). For $\beta < 0$ (soft spring) the curve tilts over to the left. Experiments confirming this hysteresis effect are reported in [6].

A first order expansion can also be developed for equation (1) which includes the damping term $c\dot{x}$. This yields [see 7]

$$\omega^2 = 1 + \frac{3}{4}\beta A^2 \pm \frac{F}{A}\left[1 - \frac{c^2A^2}{F^2}\right]^{\frac{1}{2}} \tag{3}$$

where we have assumed that the linear coefficient, k, is 1. From (3) we observe that

$$|A| < \frac{F}{c} \tag{4}$$

where F is assumed positive. A different expansion using the method of Kryloff and Bogoliuboff [8] yields

$$\omega^2 = 1 + \frac{3}{4}\beta A^2 - \frac{c^2}{2} \pm \frac{F}{A}\left[1 - \frac{c^2}{F^2}\left(1 + \frac{3}{4}\beta A^2 - \frac{c^2}{4}\right)\right]^{\frac{1}{2}} \tag{5}$$

from which an upper limit for A can be found namely

$$A^2 = -\frac{2}{3\beta}\left(1 - \frac{c^2}{4}\right) \pm \left[\frac{4}{9\beta^2}\left(1 - \frac{c^2}{4}\right)^2 + \frac{4F^2}{3\beta c^2}\right]^{\frac{1}{2}} \tag{6}$$

in which the + sign will be used for $\beta > 0$.

Comparisons of the maximum amplitude given by eqs (4) and (6) differ significantly as shown later where, in the case studied, the estimate from eq (4) is grossly misleading.

It should be stressed that these approximations do not yield any information about the effect of the initial conditions on the final solution (after the transients have died down). In other words the effect of the initial conditions on which branch 4-6 or 2-3 the solution lies is not clear from the analytic studies. Only numerically can we assess the effect of the initial conditions.

3.0 INTEGRATION METHODS

3.1 Central Difference Method

If we replace first and second derivatives in (1) at time t by central difference expressions we obtain

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2} + c\,\frac{x_{n+1} - x_{n-1}}{2\Delta t} + kx_n + \beta x_n^3 = F \sin \omega t_n$$

from which x_{n+1} can be found. Thus the step from time t to $t + \Delta t$ can be made with accuracy $0(\Delta t^4)$ since the central difference formulas are of $0(\Delta t^2)$ for the derivatives.

However this explicit method is unstable unless Δt is less than a critical value and so in practice the method is not normally used [1]. Implicit methods given below have better stability and/or accuracy.

3.2 Houbolt's Method

In this case we replace the derivatives at time $t + \Delta t$ with backward difference formulas using values at three previous points. Thus

$$\ddot{x}_{n+1} = \frac{1}{\Delta t^2}\left[2x_{n+1} - 5x_n + 4x_{n-1} - x_{n-2}\right] + 0(\Delta t^2)$$

and

$$\dot{x}_n = \frac{1}{6\Delta t}\left[11x_{n+1} - 18x_n + 9x_{n-1} - 2x_{n-2}\right] + 0(\Delta t^3)$$

These formulas and others are found in convenient form in Appendix III of Kopal's book [9].

On substituting these formulas into (1), written at $t + \Delta t$, we obtain an implicit formula for x_{n+1}, namely

$$\beta x_{n+1}^3 + B x_{n+1} = R \tag{7}$$

with accuracy $O(\Delta t^4)$ on each step, where

$$B = \frac{2}{\Delta t^2} + \frac{11c}{6\Delta t} + k$$

$$R = F \sin \omega t_{n+1} + \frac{5x_n - 4x_{n-1} + x_{n-2}}{\Delta t^2} + \frac{c}{6\Delta t}\left[18x_n - 9x_{n-1} + 2x_{n-2}\right]$$

The cubic equation (7) is solved by exact methods (see for example [10]) using Fortran's complex arithmetic ability. Since B is positive (c and k being positive) it can be seen from (7) that there is only one real root if β is positive. If β is negative (but small) there will be three real roots since in (7) the left hand side tends to $+\infty$ for $x_{n+1} \to -\infty$ and vice versa for $x_{n+1} \to +\infty$ whereas for typical values of x_{n+1} the predominant slope of the left hand side is positive. However we can expect the two roots away from $x_{n+1} \simeq \frac{R}{B}$ to be rather large since we assume β is small. Thus in our algorithm to find the root we select the one nearest to R/B.

Note that for nonlinearities which are more general than cubic a Newton-Ralphson scheme could be used on each step with first estimate equal to R/B.

3.3 A Higher Order Method

As shown above Houbolt's implicit method is accurate to $O(\Delta t^4)$ on each step as is the explicit central difference method. We now seek an implicit formula with errors of higher order than Δt^4.

This can be done by simply using terms in a backward difference approximation at time $t + \Delta t$; the error will decrease as we use more terms in the difference expression. To keep computations manageable we investigated up to ninth order schemes and, as shown later, found the eighth order scheme the most stable. This scheme replaces derivatives by

$$\ddot{x}_{n+1} = \frac{(13132x_{n+1} - 56196x_n + 110754x_{n-1} - 132860x_{n-2} + 103320x_{n-3} - 50652x_{n-4}}{2520\,\Delta t^2}$$

$$\frac{+14266x_{n-5} - 1764x_{n-6})}{} + O(\Delta t^6)$$

$$\dot{x}_n = \frac{(13068x_{n+1} - 35280x_n + 52920x_{n-1} - 58800x_{n-2} + 44100x_{n-3} - 21168x_{n-4}}{5040\,\Delta t}$$

$$\frac{+5880x_{n-5} - 720x_{n-6})}{} + O(\Delta t^7)$$

On substituting into (1) we obtain

$$\beta x_{n+1}^3 + B' x_{n+1} = R'$$

which can be solved as before for x_{n+1}. The accuracy of the integration is $O(\Delta t^8)$ on each step.

A comparison of Houbolt's and the eighth order scheme is presented in section 4.0

3.4 Starting Procedures

Houbolt's scheme requires $x(t-2\Delta t)$, $x(t-\Delta t)$ and $x(t)$ in order to determine $x(t+\Delta t)$. Thus at time $t = 0$ a special starting procedure is needed. A Taylor series scheme is an obvious choice. Firstly, on writing the equation at $t = 0$ as

$$\ddot{x}_0 + c\dot{x}_0 + kx_0 + \beta x_0^3 = 0 \qquad (8)$$

we can find $\ddot{x}_0$ from the initial conditions $x_0 = p$ and $\dot{x}_0 = q$. Then x_{-1} can be determined from the Taylor series

$$x_{-1} = x_0 - \Delta t\, \dot{x}_0 + \frac{\Delta t^2}{2} \ddot{x}_0 + O(\Delta t^3) \tag{9}$$

and x_1 from

$$x_1 = x_0 + \Delta t \dot{x}_0 + \frac{\Delta t^2}{2} \ddot{x}_0 + O(\Delta t^3) \tag{10}$$

For the next step we can now use Houbolt's scheme since we know x_{-1}, x_0 and x_1. The accuracy of Houbolt's scheme is $O(\Delta t^4)$ on each step. However the application of the expansions (9-10) limits the accuracy to $O(\Delta t^3)$. Note that the above scheme is equivalent to some authors' application of (9) followed by a central difference approximation to (8) about $t = 0$.

In our scheme, which is accurate to $O(\Delta t^8)$ on each step, we require a knowledge of $x(t-6\Delta t)$, $x(t-5\Delta t)$, ..., $x(t-\Delta t)$ and $x(t)$ in order to advance to $x(t+\Delta t)$. Thus we seek a starting scheme which determines x_{-3}, x_{-2}, x_{-1}, x_1, x_2 and x_3 to an accuracy $O(\Delta t^7)$. Our total integration scheme will then be accurate to $O(\Delta t^7)$. Note that a starting accuracy higher than $O(\Delta t^7)$ is not essential as in one cycle our scheme will yield an error

$$N \cdot O(\Delta t^8)$$

where $N = \frac{2\pi}{\omega \Delta t}$ and so the error per cycle is $\frac{2\pi}{\omega} O(\Delta t^7)$. We employ a Taylor series expansion around $t = 0$ in order to compute x_{-3}, x_{-2}, x_{-1}, x_1, x_2 and x_3 i.e.

$$x_n = x_0 + n\Delta t \dot{x}_0 + \frac{(n\Delta t)^2}{2!} \ddot{x}_0 + \frac{(n\Delta t)^3}{3!} \dddot{x}_0 + \ldots + \frac{(n\Delta t)^6}{6!} x_0^{VI} + O(\Delta t^7)$$

for $n = -3, -2, -1, 1, 2, 3$, where $\ddot{x}_0$ is found from substituting the initial conditions into the differential equation. $\dddot{x}_0$, x_0^{IV} and higher derivatives are found from continued differentiation, viz

$$\dddot{x}_0 + c\ddot{x}_0 + k\dot{x}_0 + 3\beta x_0^{2}\dot{x}_0 = F\,\omega \cos \omega t$$

determines $\dddot{x}_0$ while

$$x_0^{IV} + cx_0 + k\ddot{x}_0 + 3\beta x_0^{2}\ddot{x}_0 + 6\beta x_0\dot{x}_0^{2} = -F\,\omega^{2} \sin \omega t$$

determines x_0^{IV}, etc.

Thus we have a starting procedure of accuracy $O(\Delta t^7)$ which is consistent with our main integration scheme.

3.5 Stability

In order to test stability of schemes of higher order than Houbolt's we follow [1] to derive the growth matrix on each step assuming zero damping (c=0). The spectral radius (ρ) of the matrix indicates that the method is stable if $\rho < 1$. This quantity is computed numerically using the IMSL1B package EIGRF and is plotted in Fig. 2 for Houbolt's scheme and higher order schemes up to $O(\Delta t^9)$. It can be seen that Houbolt's scheme is unconditionally stable as expected whereas, for example, the $O(\Delta t^6)$ scheme is unconditionally unstable and the $O(\Delta t^8)$ scheme is stable provided $\Delta t/T < 0.11$ i.e. more than 10 steps/cycle are required. Since the latter condition is not a practical restriction and can easily be imposed we will use this scheme for our integration.

The higher order scheme requires more computations per step than Houbolt's by a factor of about 2. However, since a much larger step length is possible for the same accuracy, we can expect an overall increase in efficiency. The results section to follow shows a comparison of efficiency to acquire a certain accuracy. Good accuracy is important when we come to study the nonlinear problem, in which case the starting conditions will be critical in deciding which

amplitude-frequency branch the solution will settle upon. Thus the effect of the starting conditions must be computed accurately as we proceed forward in time.

4.0 RESULTS

4.1 Linear Equation

Our first test was to compare our eighth order results with results from Houbolt's scheme. The test was made on a case having higher frequency transients i.e.

$$\ddot{x} + 0.1\dot{x} + 25x = 25 \sin t$$

with $x_0 = 0.1$ and $\dot{x}_0 = 0$. It can be seen from Figs. 3A and 3B that with 64 steps/cycle the eighth order scheme gives a good prediction of the exact solution whereas Houbolt's scheme, as expected, has large errors. Even with 256 steps/cycle, Figs. 3C and 3D, Houbolt's scheme is showing larger errors than the 64 steps/cycle eighth order scheme. Thus, although the new scheme requires twice as many computations per step, the number of steps can be reduced so that the overall efficiency is about twice that of the Houbolt scheme for the same or better accuracy.

The next case, Fig. 4, is the simplest case of no damping i.e.

$$\ddot{x} + x = 0$$

It can be seen from Fig. 4A that the new scheme after 16 cycles with 32 steps/cycle is almost exact whilst Houbolt's scheme shows a marked phase shift and amplitude decrement. Even with 128 steps/cycle and after 16 cycles (Fig. 4B), a slight phase shift is still detected.

Thus the new scheme is shown to be more efficient than Houbolt's by at least a factor 2. In order to check linear accuracy in other cases a numerical integration of both the linear and nonlinear equations was simultaneously carried out in all our further computations and the linear solution was always compared to the exact. Some of these examples are covered in the next section.

4.2 Nonlinear Equation

To study the numerical behaviour in the nonlinear case we choose for our model the constants

$$\begin{aligned} c &= 0.1 \\ k &= 1 \\ \beta &= 0.1 \\ F &= 0.4 \end{aligned} \tag{11}$$

The response curve $|A|$ versus ω is shown in Fig. 5A for $0.2 < \omega < 1.4$ and in Fig. 5B for $1.18 < \omega < 1.38$. In the latter case we observe more clearly bending over of the response curve to the right. It can be seen that the Kryloff and Bogoliuboff approximation given earlier (Eq. 5) is much more accurate than the more frequently used formula (Eq. 3) in the peak region $A \approx 3$. Also shown is Duffing's approximation based on zero damping (c=0).

Our experience in generating the response curve showed that steady state solutions were obtained in typically 10-40 cycles of period $2\pi/\omega$. Away from the nonunique parts of the curve i.e. $\omega \lesssim 1.18$ and $\omega \gtrsim 1.31$ solutions were obtained in fewer iterations. In the range $1.18 < \omega < 1.31$ the branch of the response on which the solution lay depended upon the initial estimates x_0, $\dot{x}_0$. For instance with $\dot{x}_0 = 0$

several computations were made with various starting values x_0. With $x_0 = 2$ the jump from the lower branch occurred at about $\omega = 1.22$ while with $x_0 = -3.5$ the jump occurred at about $\omega = 1.32$ as shown on Fig. 5B.

Other starting values produced jumps as shown on Fig. 5C, which also shows a finer determination of the cut off jumps for non uniqueness. The corresponding phase angles relative to the driving force are given in Table 1. It was initially expected that there would be a trend of the jump values of ω with changing x_0. However as can be seen there is no apparent trend in the jump-rather there seems to be a randomness which is unexplainable. This is disconcerting particularly as there is no consistent way to produce the limits of the upper branch to the right and of the lower branch to the left. These limits in our case were determined by trial and error to be near $\omega = 1.18$ on the lower branch and near $\omega = 1.31$ on the upper branch taking $x_0 = 4$ for the lower branch and $x_0 = -3.5$ for the upper branch. In these areas near the vertical slopes solutions would often take a long time to settle on the final branch. The amplitude would tend to stick around the value on one branch for many cycles and then quite suddenly the amplitude would change to a value on the other branch. An example of this is shown in Fig. 6 for $x_0 = -3.5$ and $\omega = 1.32$.

The unstable part of the response curve shown as path 3-6 on Fig. 1 was never predicted numerically and did not cause any problems. The final amplitude would either lie on the lower or upper stable branches.

The drastic change in solution from one branch to the other can be seen by comparing Fig. 7A to Fig. 7B. These figures show converged solutions over one period of the driving force i.e. $2\pi/\omega$ for $\omega = 1.20$ and $\omega = 1.22$ respectively (with $x_0 = 2$). It can be seen that there is a large phase shift as well as an amplitude change.

Since the branch solution chosen by the numerical procedure is so dependent on initial conditions it was considered worth investigating the effect of step size Δt on the branch chosen. This is because a numerical truncation error might affect the solution to such an extent that it will jump at a different ω value for the same x_0. Thus we repeated the computations of Fig. 5C, which use 64 steps/cycle, with 128 steps/cycle. It was found that the results were identical, except for $x_0 = 10$, when the jump shifted right over to between 1.30 and 1.32 (compared to 1.18 to 1.20). However there was still no trend in results even with the smaller truncation error.

5.0 CONCLUSIONS

As a first step to studying the nonlinear response of structural components we have investigated numerically a one-degree-of-freedom nonlinear response equation. It has been shown that the multivalued response curve of amplitude versus frequency can be obtained using a numerical method of integration which is more efficient than the more standard Houbolt's scheme. The jump phenomenon from one branch of the response curve to another has been investigated and the branch chosen is found to depend upon the initial conditions but no trend can be detected in relating the frequency at which the jump occurs with initial conditions.

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Table 1. Phase Angle, ϕ, in degrees.

ω	x_o = -4	-3	-2	-1	0	1	2	3	4
1.20	56	56	56	56	163	56	56	56	163
1.22	62	62	62	169	169	169	169	169	169
1.24	62	62	62	169	169	169	169	169	169
1.26	67	67	169	169	169	169	169	169	169
1.28	73	73	169	169	169	169	169	169	169
1.30	84	84	174	174	174	174	174	174	174
1.32	174	174	174	174	174	174	174	174	174

ω	x_o = -6	-8	-10	10	8	6
1.20	56	56	163	56	163	56
1.22	62	62	169	62	169	62
1.24	62	62	169	62	169	62
1.26	169	67	169	67	169	67
1.28	169	169	169	169	169	168
1.30	174	174	174	174	174	174

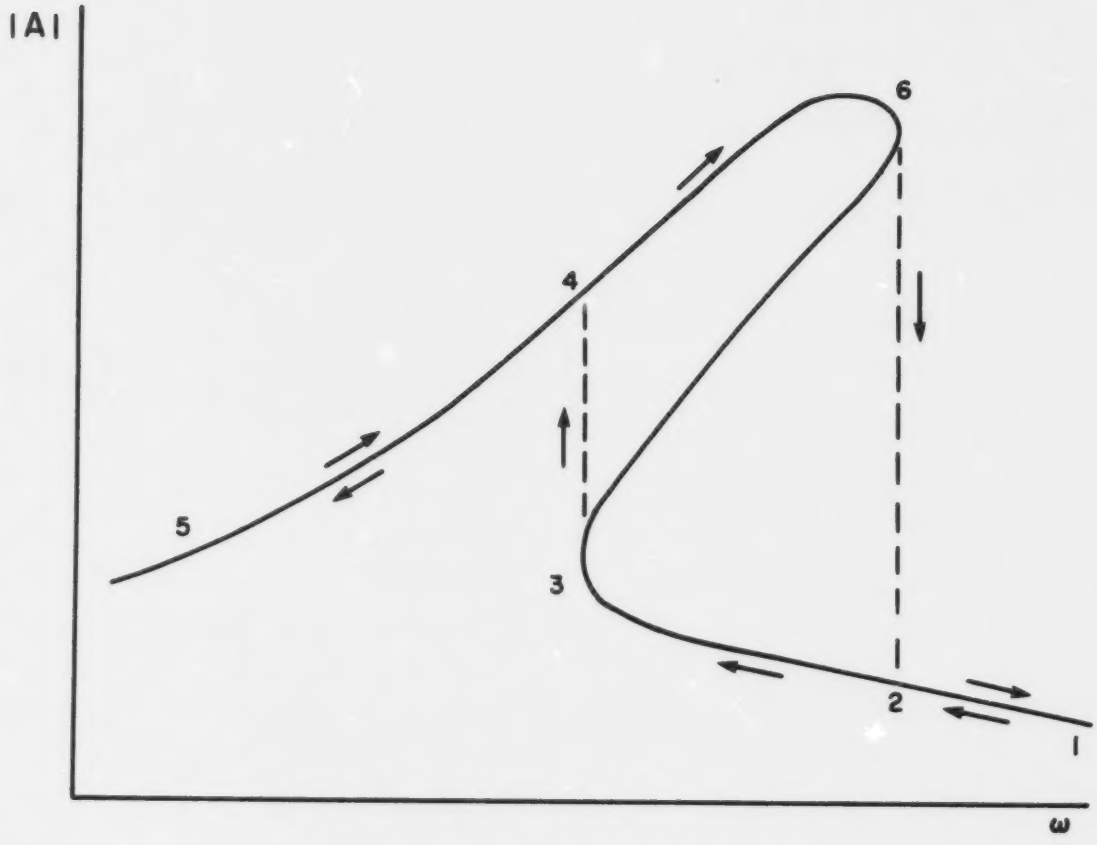

FIG. 1: SCHEMATIC OF THE RESPONSE CURVE FOR $\beta > 0$. SHOWING JUMP PHENOMENA

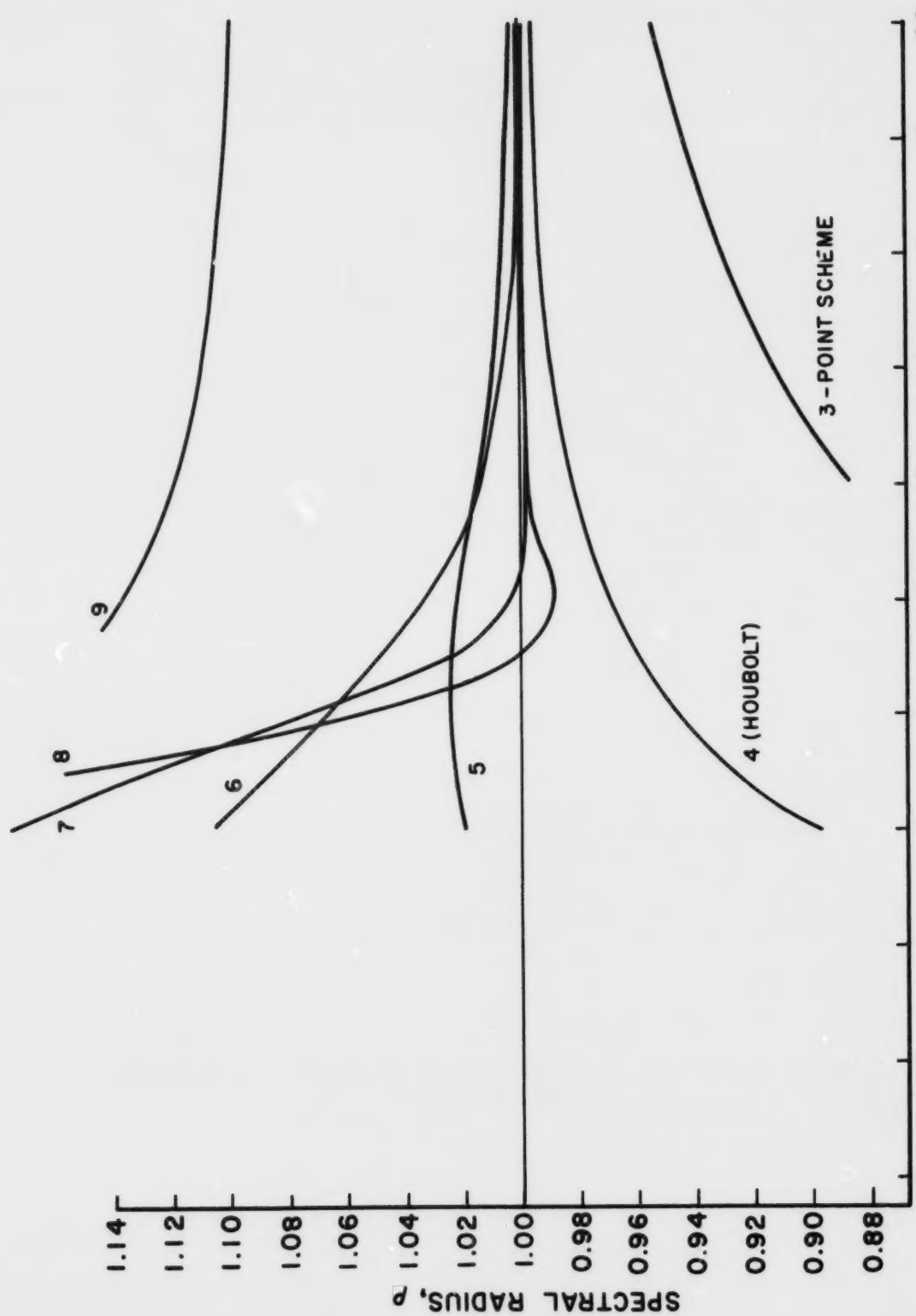

FIG. 2: STABILITY OF VARIOUS DIFFERENCE SCHEMES FOR $\ddot{X}$

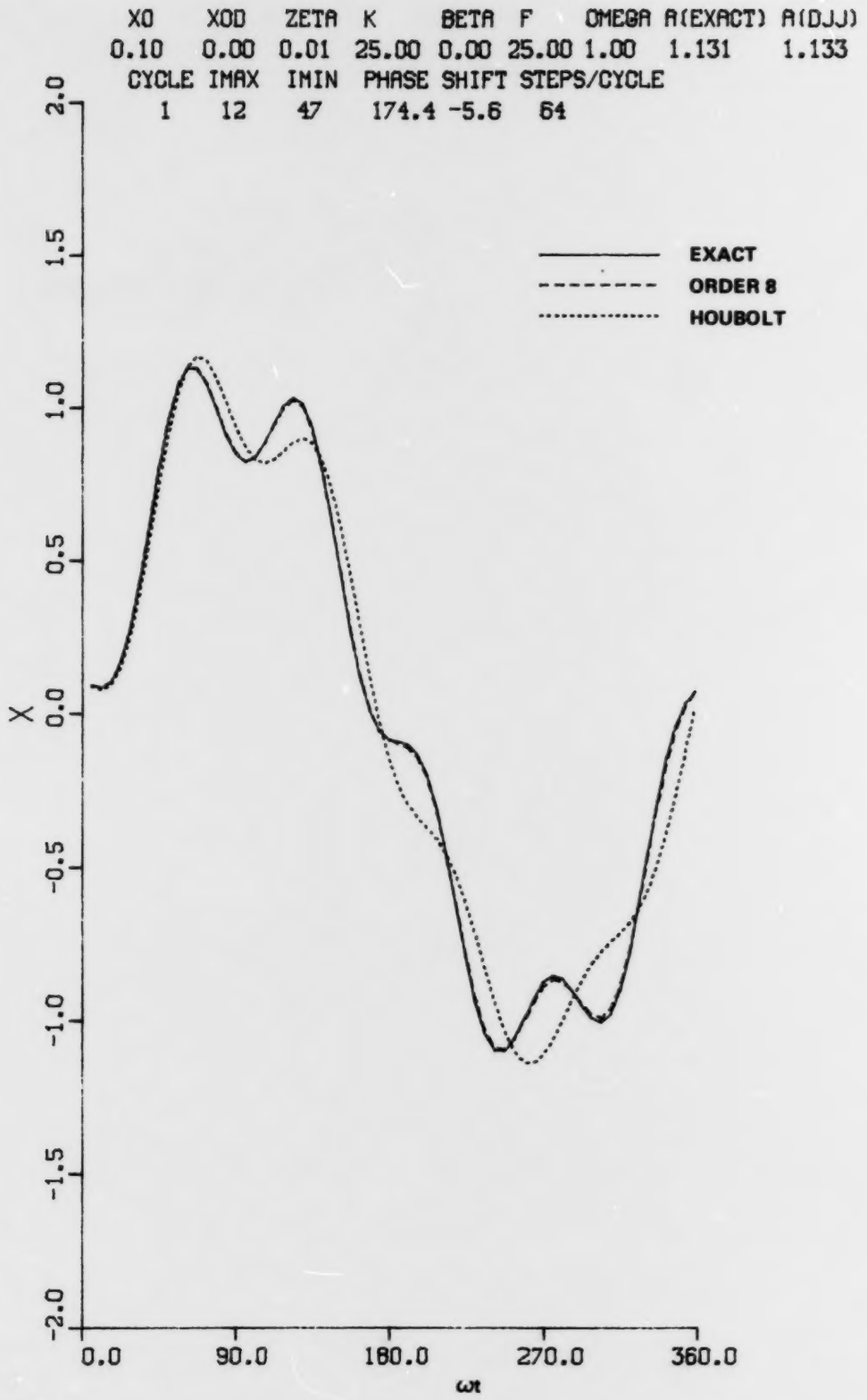

FIG. 3(a): ACCURACY OF THE EIGHTH ORDER SCHEME WITH 64 STEPS/CYCLE

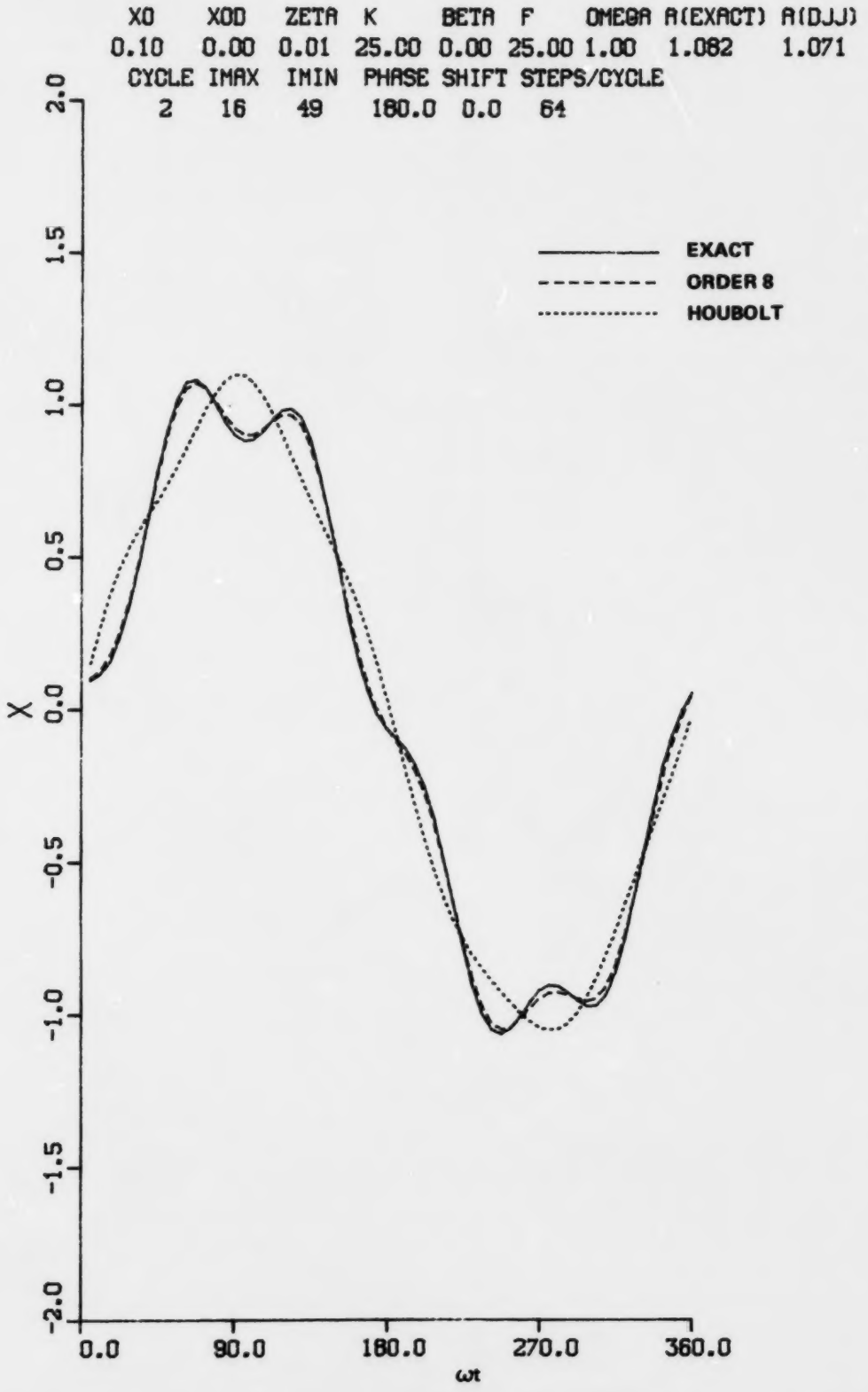

FIG. 3(b): ACCURACY OF THE EIGHTH ORDER SCHEME WITH 64 STEPS/CYCLE

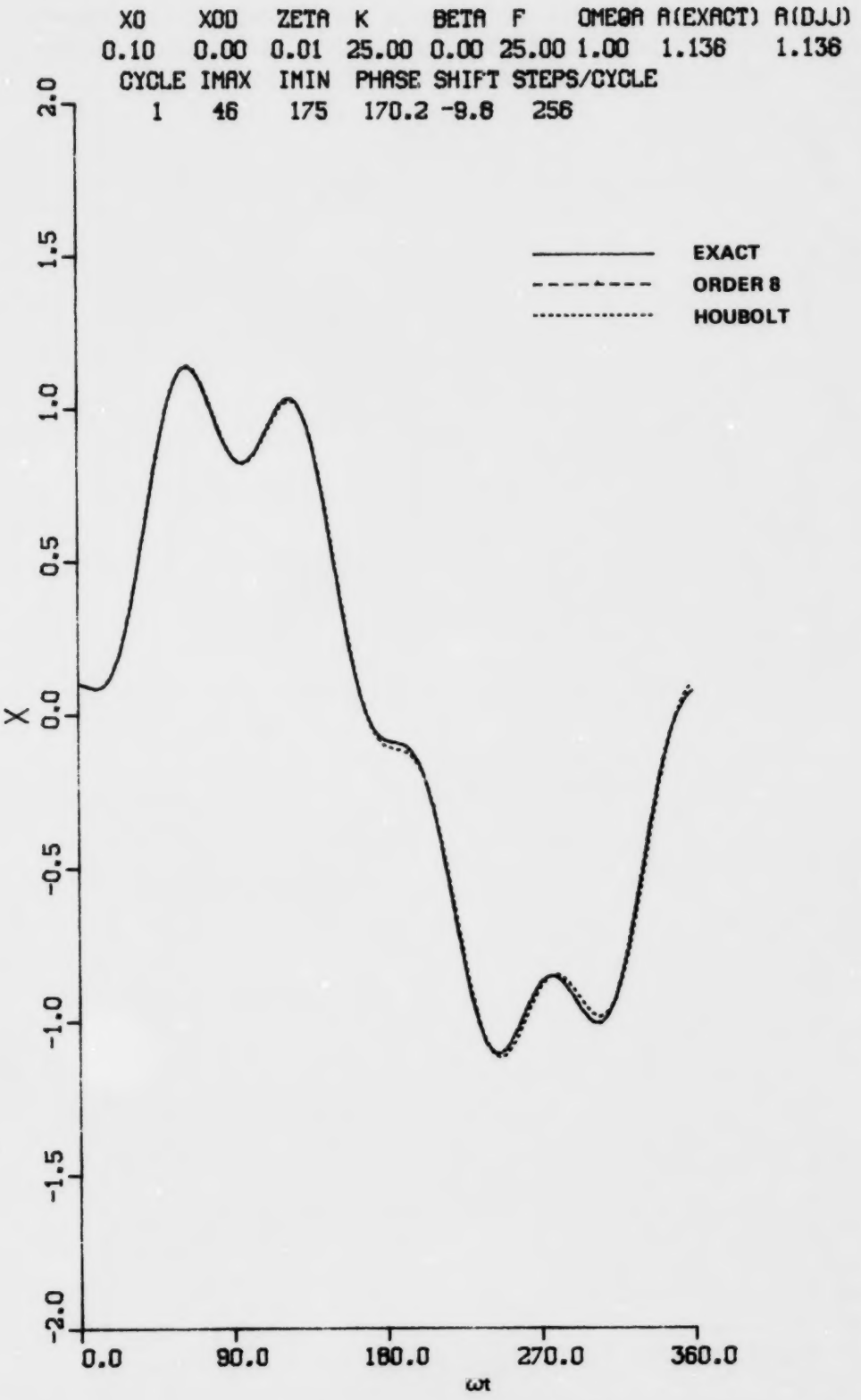

FIG. 3(c): ACCURACY OF HOUBOLT'S SCHEME WITH 256 STEPS/CYLE

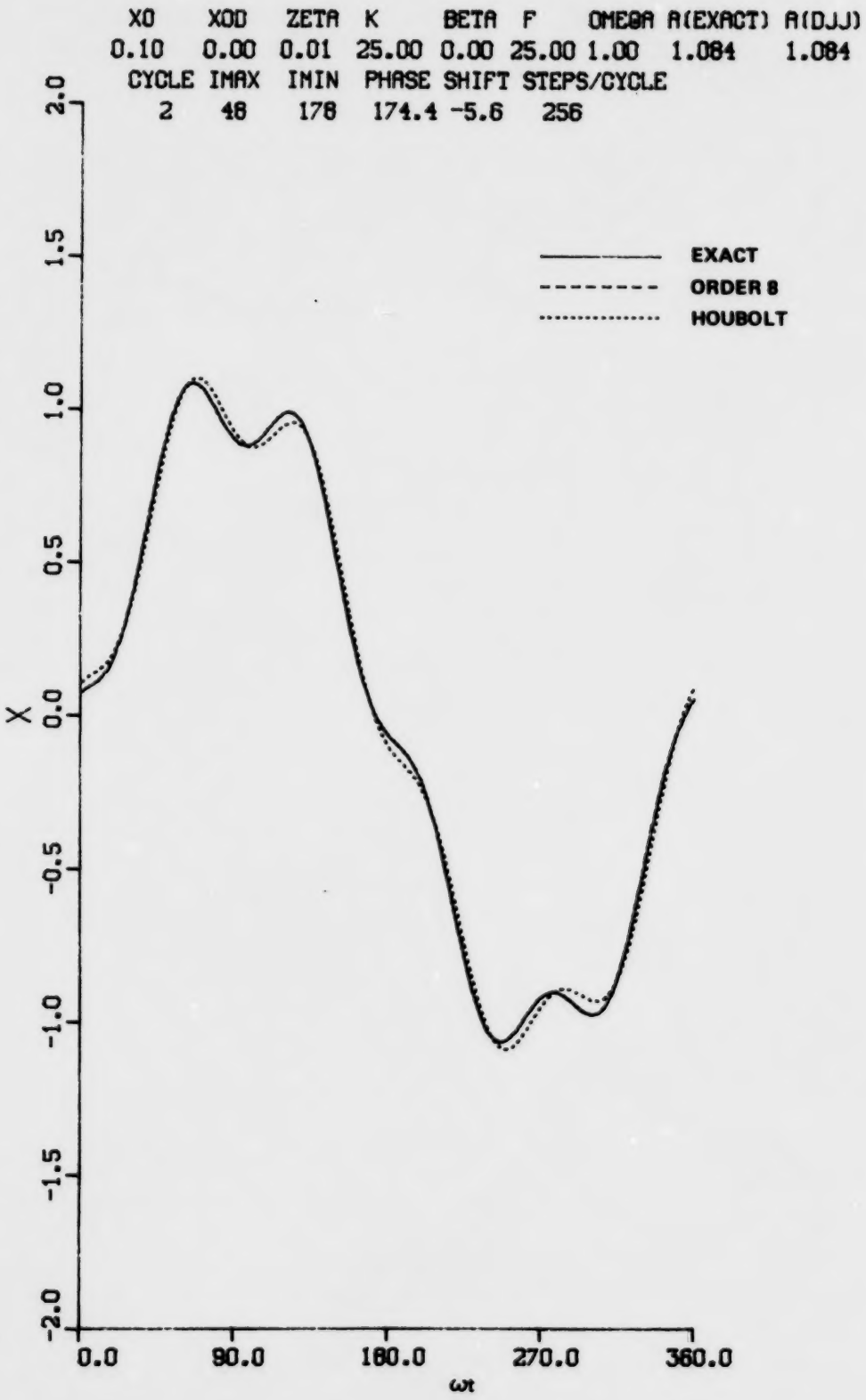

FIG. 3(d): ACCURACY OF HOUBOLT'S SCHEME WITH 256 STEPS/CYCLE

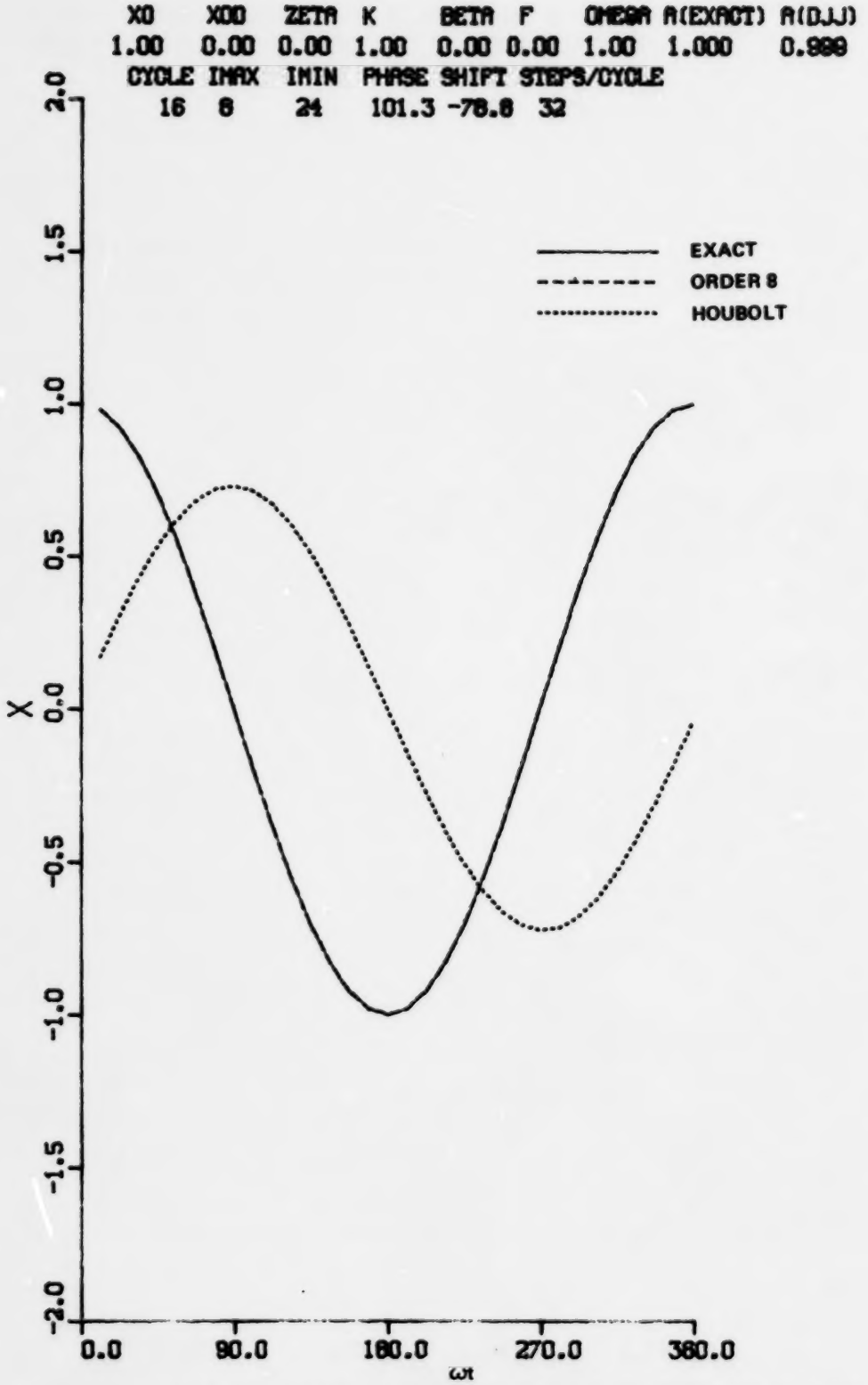

FIG. 4(a): ACCURACY OF EIGHT ORDER SCHEME FOR SIMPLE HARMONIC WITH 32 STEPS/CYCLE

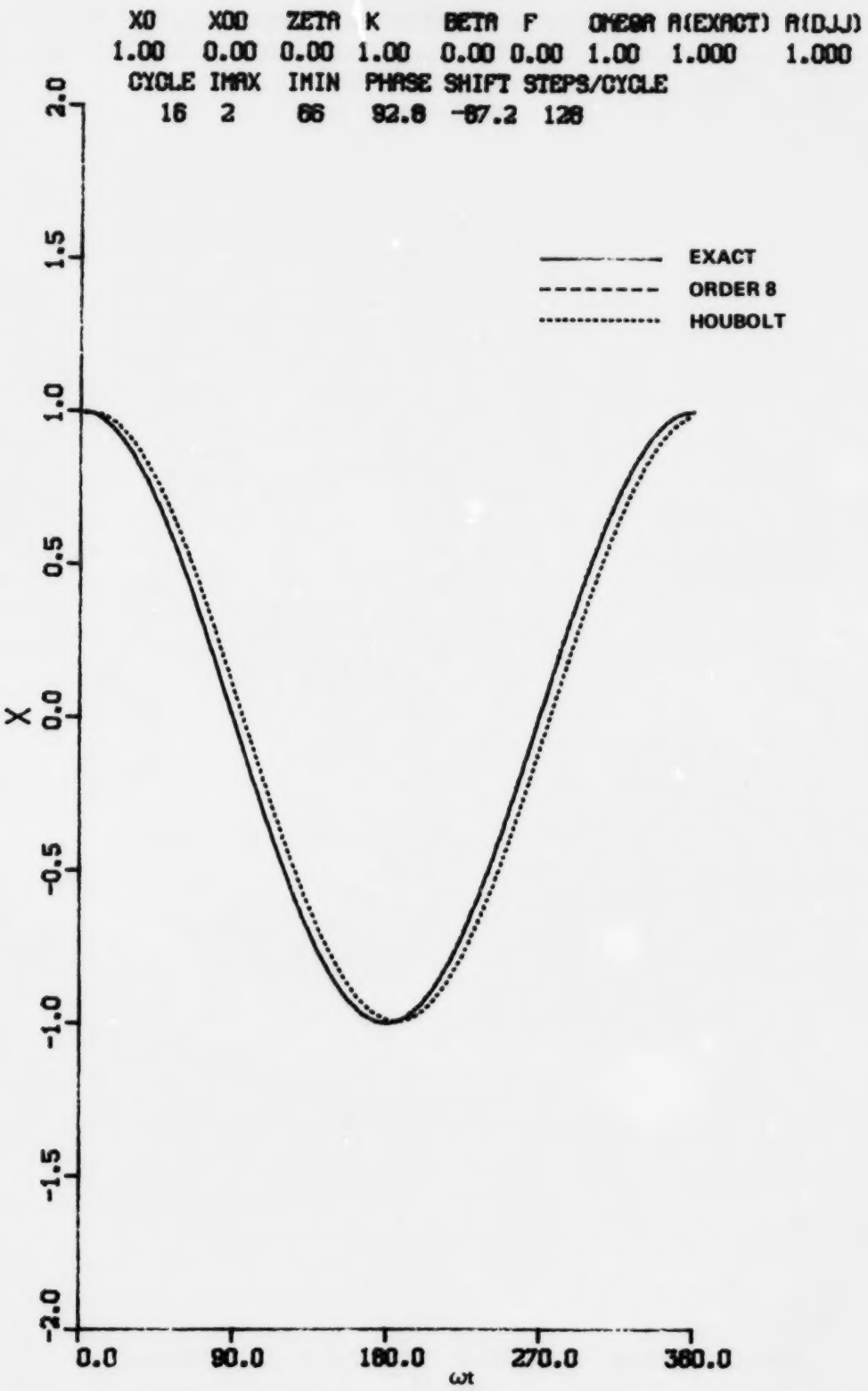

FIG. 4(b): ACCURACY OF HOUBOLT'S SCHEME FOR SIMPLE HARMONIC MOTION WITH 128 STEPS/CYCLE

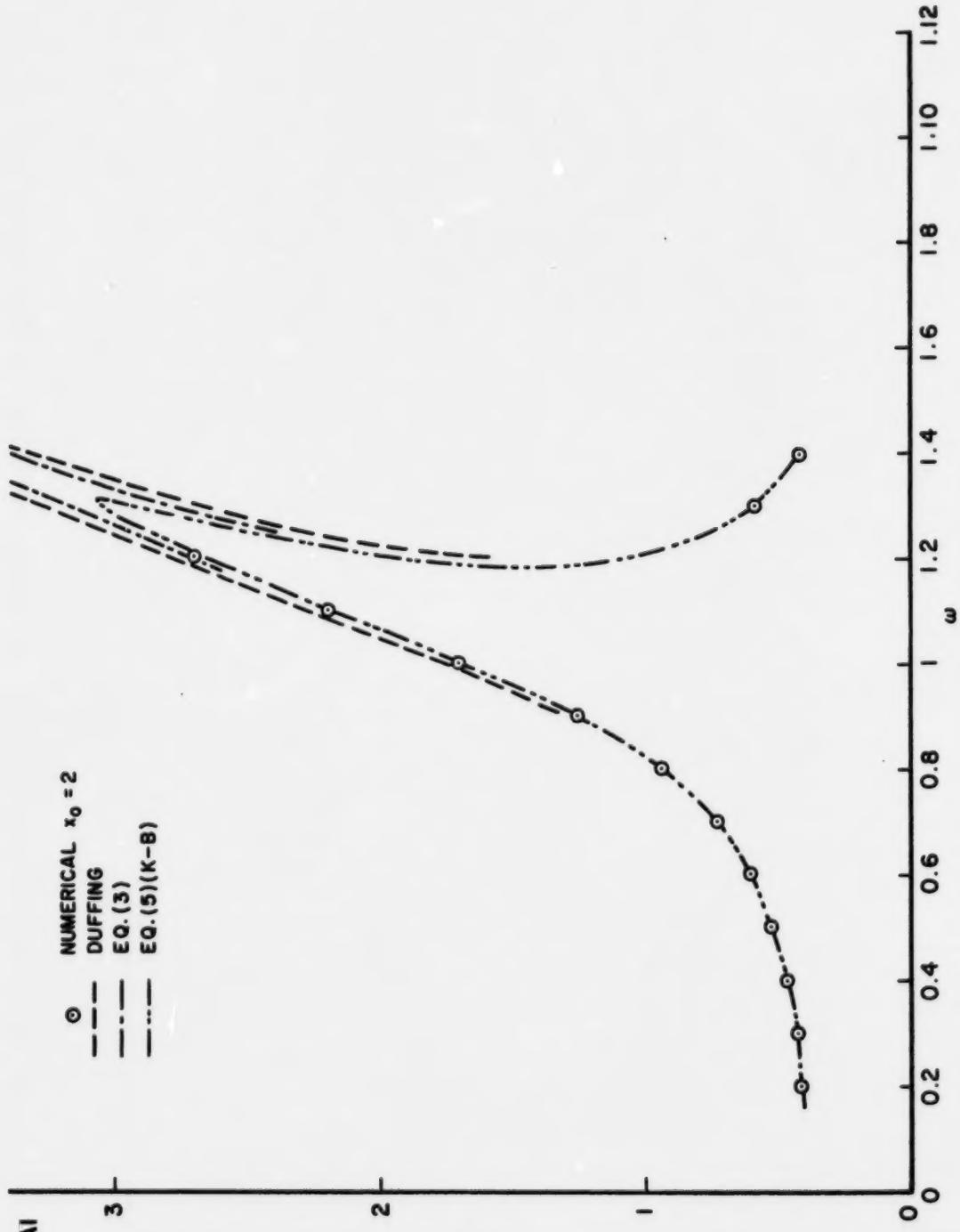

FIG. 5(a): RESPONSE CURVES FOR CONDITIONS (11)

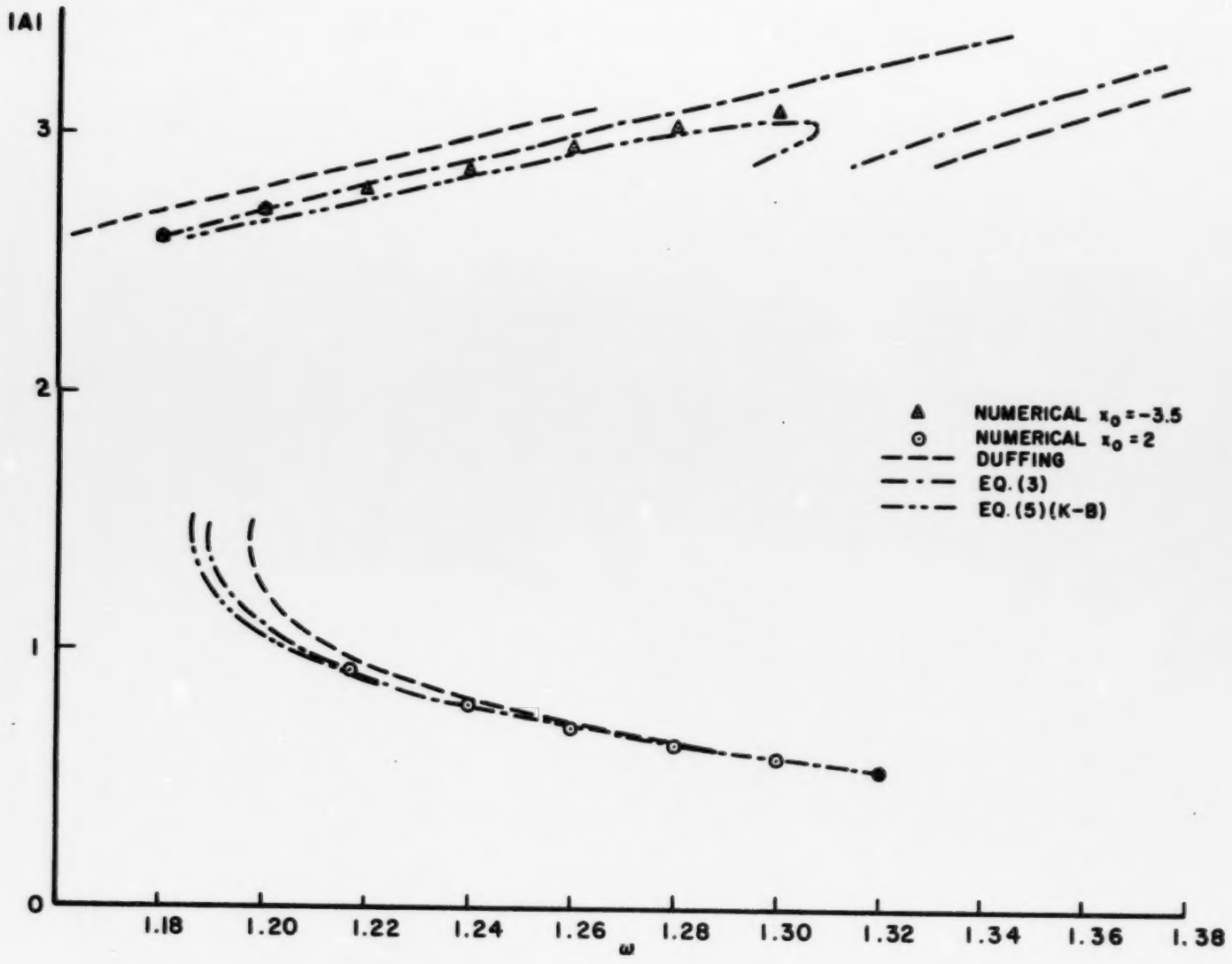

FIG. 5(b): RESPONSE CURVES FOR CONDITIONS (11) MAGNIFIED IN THE JUMP REGION

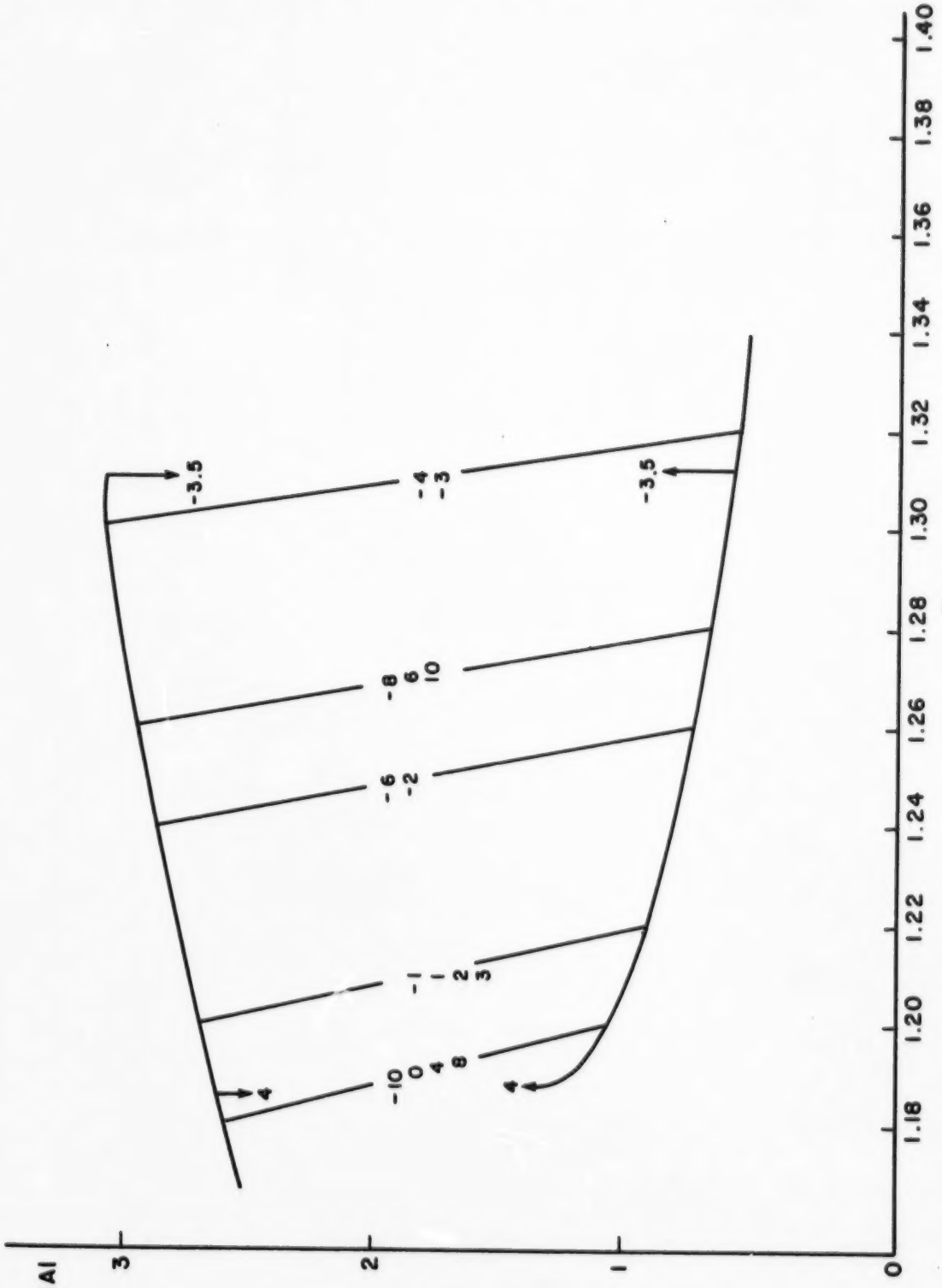

FIG. 5(c): JUMP POSITION FOR VARIOUS x_0 WITH $\dot{x}_0 = 0$

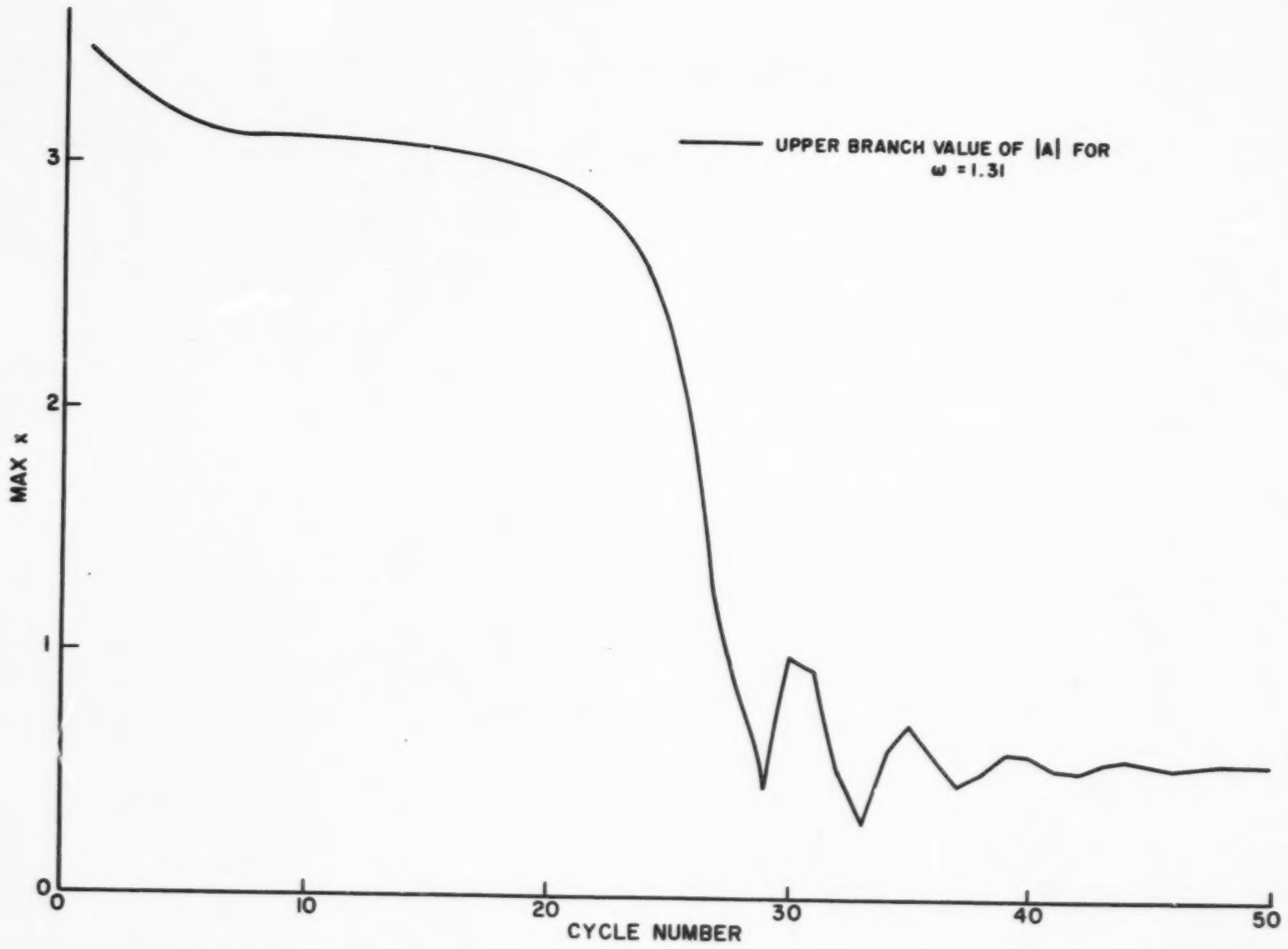

FIG. 6: SLOW CONVERGENCE NEAR THE UPPER BRANCH RIGHT HAND LIMIT.
$x_0 = -3.5$, $\omega = 1.32$

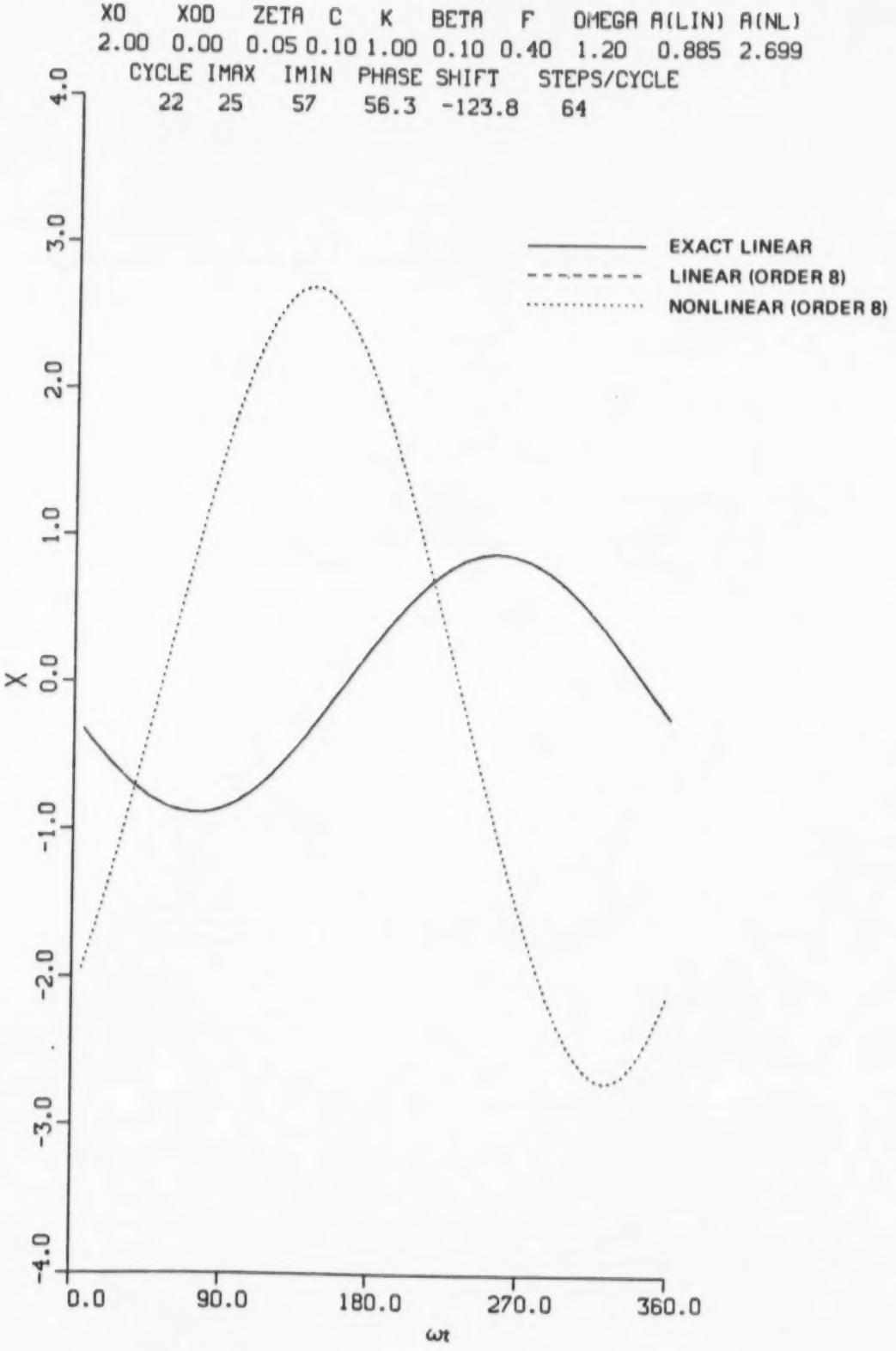

FIG. 7(a): SUDDEN CHANGE IN SOLUTION ACROSS THE JUMP

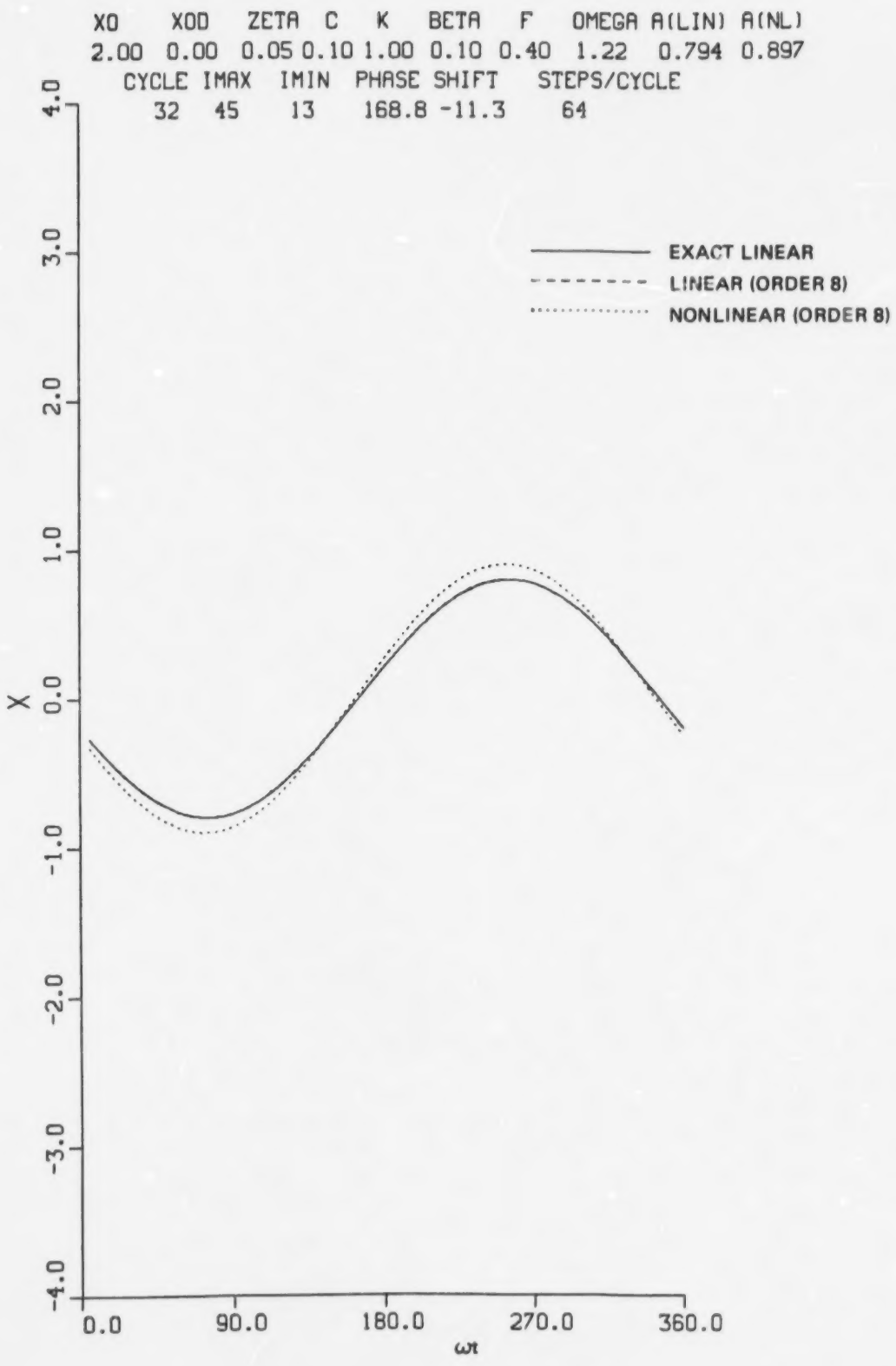

FIG. 7(b): SUDDEN CHANGE IN SOLUTION ACROSS THE JUMP

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NOTES

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DESCRIPTORS (KEY WORDS)/MOTS-CLÉS

1. Numerical analysis
2. Structural analysis
3. Duffing's equation

(14)

SUMMARY/SOMMAIRE

As part of the study of the nonlinear response of structural components we investigate in this report a single degree-of-freedom nonlinear response equation usually known as Duffing's equation. Previously the response curves have been investigated analytically by many authors but there seems to have been very little numerical work done in the nonlinear cases. It is shown that a numerical solution is feasible and that some of the standard analytical approximations are not accurate in the peak region of the response curve.

(15)